Linear systems – Resit exam

Resit exam 2018–2019, Tuesday 9 July 2019, 9:00 – 12:00

Instructions

1. The use of books, lecture notes, or (your own) notes is not allowed.
2. All answers need to be accompanied with an explanation or calculation.

Problem 1 (10 points)

Solve the initial value problem

$$\dot{x}(t) + \frac{t}{1+t^2}x(t) = (1+2t^2)\sqrt{1+t^2}, \qquad x(0) = 2.$$

Problem 2 ($4 + 8 = 12$ points)

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= x_1(t) - x_1(t)x_2(t),\\ \dot{x}_2(t) &= -x_1^3(t) + u(t).\end{aligned}$$

(a) Take the nominal input $u(t) = \bar{u} = 1$ for all $t \geq 0$. Show that $\bar{x} = [\,1\ \ 1\,]^{\mathrm{T}}$ is the unique equilibrium point corresponding to this input.

(b) Linearize the nonlinear system around $\bar{x} = [\,1\ \ 1\,]^{\mathrm{T}}$ and $\bar{u} = 1$.

Problem 3 ($4 + 10 + 6 = 20$ points)

Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t),$$

with state $x(t) \in \mathbb{R}^2$, input $u(t) \in \mathbb{R}$, and where

$$A = \begin{bmatrix} -7 & -3 \\ 22 & 10 \end{bmatrix}, \qquad B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

(a) Is the system controllable?

(b) Find a nonsingular matrix T and real numbers α_1, α_2 such that

$$TAT^{-1} = \begin{bmatrix} 0 & 1 \\ \alpha_1 & \alpha_2 \end{bmatrix}, \qquad TB = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

(c) Use the matrix T from problem (b) to obtain a state feedback of the form $u(t) = Fx(t)$ such that the closed-loop system matrix $A + BF$ has eigenvalues -3 and -2.

Problem 4 (14 points)

Determine all values of $a \in \mathbb{R}$ and $b \in \mathbb{R}$ for which the linear system

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -b & -a & -2 & -a \end{bmatrix} x(t) \tag{1}$$

is asymptotically stable.

Problem 5 (8 + 4 + 4 = 16 points)

Consider the matrices

$$A = \begin{bmatrix} -1 & -1 \\ -2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & 2 \end{bmatrix}. \tag{2}$$

(a) Determine e^{At}.

In the remainder of this problem, consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t),$$

with $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$ and the matrices A, B, C given by (2).

(b) Is the linear system externally stable?

(c) Determine the transfer function of the linear system.

Problem 6 (8 + 10 = 18 points)

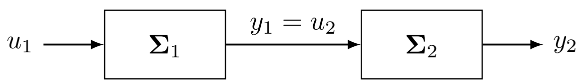

Figure 1. Cascade interconnection of two systems

Consider two linear systems

$$\mathbf{\Sigma}_i: \quad \dot{x}_i(t) = A_i x_i(t) + B_i u_i(t), \quad y_i(t) = C_i x_i(t)$$

for $i \in \{1, 2\}$ and their cascade interconnection given by $u_2(t) = y_1(t)$ as given in Figure 1. Then, the dynamics of the interconnection can be described as

$$\mathbf{\Sigma}: \quad \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ 0 \end{bmatrix} u_1(t).$$

Assume that A_1 and A_2 have no eigenvalues in common (i.e., $\sigma(A_1) \cap \sigma(A_2) = \emptyset$) and that the interconnection $\mathbf{\Sigma}$ is controllable.

(a) Show that the matrix pair (A_1, B_1) is controllable.

(b) Show that

$$\operatorname{rank} \begin{bmatrix} A_2 - \lambda I & B_2 T_1(\lambda) \end{bmatrix} = n_2,$$

for all $\lambda \in \sigma(A_2)$, where n_2 is the state space dimension of $\mathbf{\Sigma}_2$ (i.e., $x_2(t) \in \mathbb{R}^{n_2}$) and T_1 is the transfer function of $\mathbf{\Sigma}_1$ given as $T_1(s) = C_1(sI - A_1)^{-1} B_1$.

Hint. Use the Hautus test.